Enactivism in mathematics education: moving toward a re-conceptualization of learning and knowledge

Jérôme Proulx, Elaine Simmt

Abstract: *The paper explores three topics: interpretations and the world of significance; problem solving and problem posing; and knowledge as acquisition and knowledge as being. Through these topics we illustrate the distance of enactivism from constructivism and the various extensions of it. After having discussed each theme and the issues they raise, we will highlight our appreciation of learning, knowledge, teaching and curriculum, proposing conceptualizations that offer paths of understandings and research.*

Riassunto: *L'articolo approfondisce tre temi: le interpretazioni e il mondo di significato, problem solving e problem posing, la conoscenza. Attraverso questi temi ci illustrano la frattura tra enattivismo e il costruttivismo e le varie estensioni di esso. Dopo aver discusso ciascun tema e le questioni che sollevano, metteremo in luce l' apprezzamento per l'apprendimento, la conoscenza, l'insegnamento e il curriculum, portando a concettualizzazioni che offrono percorsi di comprensione e di ricerca.*

Keyworks: *Enactivism, Mathematics Education, Learning, Knowledge.*

Introduction – contextualizing the work

Learning and knowledge, learners and knowing are the primary concern of educators. In mathematics education constructivism and constructivist research have had a privileged place in mathematics education to make sense of these phenomena for decades. Ever since Piaget used counting and sets to explore children's cognitive development, Western views of mathematical knowing largely have been written from a constructivist perspective. It isn't surprising that we ourselves have been immersed in the constructivist paradigm from the beginning of our tertiary education, having been mentored by internationally renowned mathematics education researchers immersed

in those discourses (Bednarz, Kieren) and have continued to be influenced by that discourse in our research programs (Bednarz, Proulx, 2011; Proulx, 2006; Simmt, 2006). At the same time, because of our ongoing efforts to understand learning and teaching, we have been intrigued with the body of work that is referred to as "enactivist" (Maturana, 1987, 1988a, 1988b; Maturana, Varela, 1992; Varela, 1996, 1999; Varela, Thompson, Rosch, 1991). It is our contention that this literature, albeit not completely separate from constructivism, offers significantly new and important ideas that have the potential to alter our views of learning, knowledge, and education (Proulx, 2008a, 2008b 2010; Proulx, Simmt, Towers, 2009). In this paper, we articulate and explore three generative themes that we have come to understand through the enactivist paradigm, and discuss what these mean for us as mathematics education researchers who study learning and knowledge, teaching and curriculum[1]. The three themes we address are: (1) interpretations and bringing forth a world of significance; (2) problem solving and problem posing; and (3) knowledge as acquisition and knowledge as being. Through these themes we illustrate the departure of enactivism from constructivism and the various extensions of it.

We explore each of the three themes separately, in order to explore their meaning and ramifications. After having discussed each theme, we comment on how these themes, and the issues they raise, alter and transform our appreciation of learning, knowledge, teaching and curriculum, leading to conceptualizations that offer paths of understandings and avenues for research. Because this is *our story* with/in enactivism, we contextualize our views by discussing our entry into the enactivist literature and its ideas before we elaborate on each theme. We do this from our own positioning, that is, as mathematics education researchers.

Contextualizing our entry into enactivism

Discussing issues of enactivism and establishing distinctions between these and constructivist ones can be problematic, for a number of interconnected reasons. First, both theories or "world views" have similar origins in non-objectivist thinking and emerge from the writing of scholars who have collaborated together. It is difficult to assert which scholars are best identified as constructivists and which are best identified as enactivists. Humberto Maturana is a good example. Whereas we view him as a

foundational contributor to the enactivist world-view, many constructivists see Maturana as "one of them" (*Constructivist Foundations*, http://www. univie.ac.at/constructivism/journal/index.html). Heinz von Foerster is another example, whereas his writings are viewed as part of the constructivist discourse, and he shares many ideas with Ernst von Glasersfeld, we see his work as significantly related to the enactivist literature. Similar things could be said of Francisco Varela, Gregory Bateson, and Maurice Merleau-Ponty. As educational researchers who have developed their work within enactivism, we too could be seen under that same lens as we have written and presented at different times under each world view. Thus, we acknowledge that enactivism is not mutually exclusive of constructivism *per se*. Rather we understand it to place different and more explicit emphases on various issues (three of those we explore in this article) and that different and differential emphases lead to different implications for educational issues.

A related second difficulty is that neither theory is definitively defined. Through the years, many versions of constructivism have been posited (e.g. radical, social, pragmatic, psychological, pedagogical, etc.; Larochelle, Bednarz, Garrison, 1998; Noddings, 1990; Philipps, 2000; Steffe, Gale, 1995; Watzlawick, 1984). Some versions are very close to the original writings (of Piaget, e.g.) whereas others stray farther away as theorists address perceived deficiencies in previous elaborations of constructivism. Thus, when someone mentions constructivism, there is often a felt-need to qualify the version of it – "Are you talking about the "social" version of constructivism, its "radical" version?". To settle this issue for ourselves, we explicitly address von Glasersfeld's constructivism that was grounded in epistemological considerations drawn extensively from Piaget's writings. We also favour this view, because radical constructivism has been widely referred to in mathematics education research (Glasersfeld, 1991). Enactivism, in contrast, has never really been established as a specific world view in the educational research and literature. Consider the two theorists who most influenced *our* view of enactivism, Humberto Maturana and Francisco Varela. Neither has used, to our knowledge, the expression "enactivism" or "enactivist". Hence, it is not clear who coined the expression "enactivism," although Varela sporadically used the expressions "enactive" and "enaction" a number of times (Varela, Thompson, Rosch, 1991; Varela, 1996), before using it quite extensively in his *Ethical Know-How* (Varela, 1999). We also note that Maturana never used the expression; the categorization of his

work as "enactivist" could be disputed, and may be by Maturana himself. Discussing "the" enactivist world view can thus be problematic, since there are no original enactivist texts to which all researchers refer (Maturana, Varela's *Tree of knowledge*, and Varela, Thomspon and Rosch's, *Embodied Mind* may be the seminal texts for many, but not for all enactivist researchers). As for us, we trace our work in enactivism to the mathematics education research group working with Tom Kieren at the University of Alberta, Canada in the 1990s. In their work, they explicitly use the expressions "enactivism" and "enactivist" (Davis, 1995, 1996; Reid, 1996), and refer to similar authors as we do.

Finally, a third difficulty emerges as one tries to identify which texts belong to the enactivism discourse, specifically as it is taken up in education. From one author to the next who claim to use enactivism, we see quite varied references to scholars, articles, books, chapters, etc., that must be, for any particular author, representative of the enactivist literature. Obviously there are intersection points among the texts and they overlap, but there are also important differences that lead authors to focus on very different aspects that they might claim to be central (or peripheral) to enactivism. Whereas in the constructivist literature discussions about concepts that make distinctions (e.g. viability, experience, truth) are readily available, the same cannot be said about enactivism. Although this poses a challenge when entering discussions with others who identify with enactivism, we believe this also opens up possibility, since there is an opportunity to push the collective thinking in the field, and avoid the reification of enactivism as one specifically bounded discourse (like what happened, e.g., with constructivism; Bednarz, Proulx, 2011).

Enactivism offers, for us, a way of continuously developing a non-objectivist view of the world, and a view of knowledge issues that can be used productively in mathematics education in particular. The relationships and distinctions we trace in this article are not to be seen as the "things-in-themselves", as the *ding an sich*, but as issues that the enactivist literature has occasioned for us. In that sense, and significantly, the elements and issues we outline and address here are not necessarily explicitly outlined in those works and texts we refer to. Rather, those texts made it possible, they made possible the distinctions we make and explore. These distinctions have led to the continuous evolution of *our* world view, as mathematics education researchers; a view of the world that we refer to as enactivist. We now explore each of the three themes as we nuance our conception of

enactivism. But, first, we offer the following vignette as a means of grounding our discussion.

Vignette on mathematics knowing

Imagine a mother and her 10 year old daughter working on a mathematics task together. They have been given a box of dominoes (2x1 tiles) and asked how many different arrangements exactly two units wide can be created from a specific number of dominoes. The facilitator demonstrated the task by illustrating that one tile could only be laid in one direction (vertically), two tiles could be laid in two ways (either standing vertically next to each other or laid horizontally) and be two units wide, and three tiles could be arranged in three different ways while still satisfying the constraints of being two units wide. The mother and her daughter drew possible tilings of four and five tiles respectively on a piece of paper, attending to the arrangements (see Figure 1). Across the table are a father and his daughter working in response to the same prompt, but keeping much different records. On their working paper is a table that lists the number of tiles used and the number of different arrangements made for the particular number of tiles (Figure 2). An observer of the parents and children might wonder about the mathematics these two pairs are doing and how it has come to be that the artefacts of their doing look so different (c.f. Simmt, 2000).
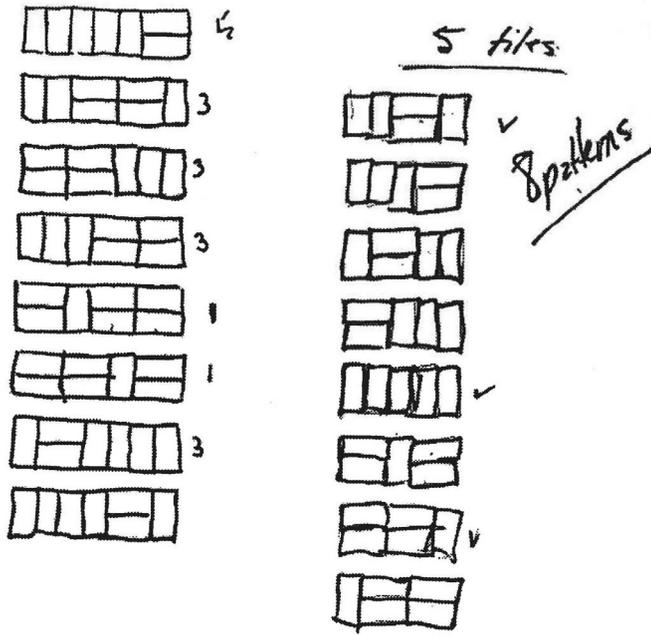
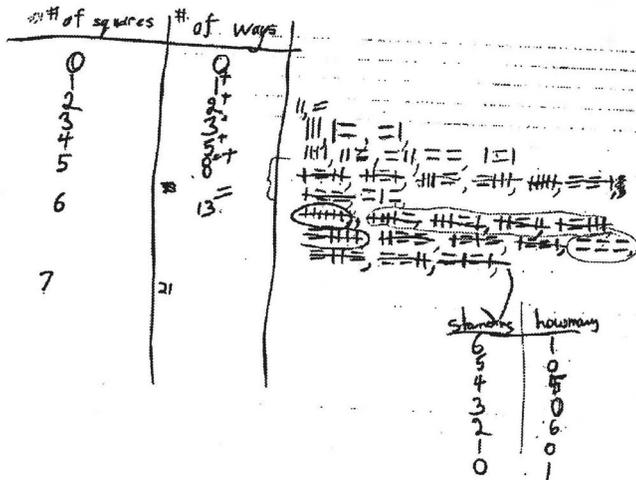Figure 1. Mother and daughter records of tilings

Figure 2. Father and daughter records of tilings

Theme 1: Interpretation and bringing forth a world of significance

Issues of "interpretation" of reality differ between enactivist and constructivist thinking. A main concept in constructivist thinking used to explain a person's response to a trigger or prompt is that of viability. Constructivists substituted realists' notions of truth and existence with that of viability, a concept closely aligned with the notion of "fit" which comes from Darwinian evolutionary theory.

> Constructivism goes back to Vico, who considered human knowledge a human construction that was to be evaluated according to its coherence and its fit with the world of human experience, and not as a representation of God's world as it might be beyond the interface of human experience. Constructivism drops the requirement that knowledge be 'true' in the sense that it should match an objective reality (von Glasersfeld, 1992, 3).

This leads to the perspective that there exist a number of viable interpretations of the world, each knower developing one that fits within his or her functioning of the world. In constructivist thought, interpretations are not said to be made in an arbitrary fashion, but are made on the basis of invariants or constants one finds in the world and attempts to make sense of; invariants that are found, for example, through multiple experiences, repetition, patterning in that world, etc. In relation to those invariants and constants, one constructs one's vision of the world, one's reality.

Hence, from a constructivist perspective of the actions of the parent-child pairs in the vignette, we understand the very distinct actions and outcomes of their thinking as different but viable interpretations of the micro-world that the tiles and the constraints of the prompt present. The invariants in the materials, the constraints, and the task (as posed) were taken up differently by each pair (and each individual in each pair). From the observer's perspective there was not (in an objective way) a reality that consisted of the functional relationships in the father and daughter's work or the geometry of the mother and daughter pair but both interpretations of the task were viable. In that way there is no objective reality accessible to the parent-child pairs; there is only their personal interpretation of the invariants in the micro-world afforded by the prompt and material.

In the case of enactivism, coming to know (or knowing) in a situation is not so much about the invariants within the environment, but about the coordination of the knower and the environment. The focus on invariants and what the knower can construe from the environment sets the constructivist discourse apart from the enactivist perspective. In constructivist discourses "cognition is based on, and drawn out of experience in an environment, [hence] the interactional dynamics with the environment are only considered from the point of view of the individual" (Sumara, Davis, Kieren, 1996, 156). But from an enactivist perspective knower and known, both organism and environment, co-evolve in a constant process of becoming.

In the case of the mathematics of the parent-child pairs in the vignette, they posed what was relevant in their domain and created what we observe as two very different mathematical worlds of significance. The mother and daughter brought forth a geometric space, one in which the physical arrangement of the tiles was significant. (Note the drawings represented not only the arrangement of the tiles but the their physical dimension as 2x1 tiles, as well). This is in contrast to the father-daughter pair who was very focused on counting how many arrangements there were for a given set of tiles and subsequently on the functional relationship between the number of tiles and the number of arrangements. (Note the highly schematicized drawings of the tiles). Both pairs were interested in determining if they had all of the tiles for a given set, but whereas the mother and daughter considered the reflections and rotations of the arrangements, the father and daughter reflected back to the numbers recorded in their table to see if they could predict if they had all of the arrangements.

As these knowers brought forth their worlds of significance, the environment in which they were coupled was also transformed, albeit according to its own dynamics. For the father-daughter pair moving individual dominoes about on the table transformed to deliberate moves of single vertical tiles and pairs of horizontal tiles as the record of their actions grew and they modified their coding and means of communicating with each other. This transformation of the environment resulted in them working with objects that co-emerged with their own activity. The tiles gave way to the table of values as mathematically interesting to the pair. In contrast the mother-daughter pair could be observed as acting geometrically, as they sought out mirror images (reflections) and other transformations of the arrangements. The mathematical world they brought forth was of a differ-

ent order, and about different mathematical concepts (geometry, mirroring, spatial representation *versus* quantitative, functional).

Capra (1996) insists that Maturana and Varela offer a theory of co-evolution. There is thus no fixed state for the interpreter to interpret, no invariants or constants that *are*, since both interpreter and state are in flux, influencing each other in the ongoing process of living. In enactivism, knower and known co-emerge with and in the interaction. Hence environment is not static and therefore cannot be interpreted with its regularities. The environment is being defined while defining the knower in their interaction.

> The actions of an animal and the world in which it performs these actions are inseparably connected. [...] What is perceived appears inseparably connected with the actions and the way of life of an organism: cognition is, as I would claim, the *bringing forth of a world*, it is embodied action (Varela, Poerksen, 2004, 87).

With enactivism the view of viability of interpretations gives way to the notion of the knower being brought forth as he or she brings forth a world. Moment by moment we bring forth our worlds of significance. As observers we note a coupled relationship between the person and the object (Maturana and Varela, 1992). The knower and the known co-dependently arise through mutual specification (Varela, Thompson and Rosch, 1991). Hence rather than learners interpreting the world in multiple ways, learners bring forth different and distinct worlds of significance with their knowing. Maturana explains this well in an interview with Simon (1985):

> Systems theory first enabled us to recognize that all the different views presented by the different members of a family has some validity, but systems theory implied that there were different views of the same system. What I am saying is different. I am *not* saying that the different descriptions that the members of a family make are different views of the *same* system. I am saying that there is no one way which the system is; that there is no absolute, objective family. I am saying that for each member there is a different family, and that each of these is absolutely valid (Maturana, in Simon, 1985, p. 36).

This is not an issue of interpreting "something" or of giving meaning to a situation: because the knower is coupled with/in this situation, where

both are co-defining one another in the relationship. The knower poses what is relevant in his or her domain and creates what an observer distinguishes as a response to the prompt. Knowers act in multi-verse, bringing forth worlds, rather than interpret the uni-verse in multiple ways. Hence, a distinction that we read from both Maturana and Varela is that these are not multiple interpretations of the problem but rather that learners bring forth different and distinct worlds of significance with their knowing.

The issue we raise about bringing forth a world of significance offers a way of conceiving about the enmeshment of knower and known. This issue is not about interpreting a world, but about bringing it forth. With this distinction emerges an issue with how we have traditionally thought about cognitive acts as acts of problem solving. We now reflect on how to think about what constitutes a problem and the status of problems in a person's world if that world comes into being, is brought forth, through the knower's actions.

Theme 2: Problem solving and problem posing

For Varela (1996; Varela *et al.*, 1991), problem solving would imply that problems are already in the world, lying "out there" waiting to be solved, independent of us as knowers. Varela explains that because of what we are biologically, historically, socially, culturally, etc., because we are coupled with the environment, and because we and our world co-dependently arise, we do not encounter problems to solve but specify the problems that we encounter through the meanings we make of the world in which we live. We confront the environment and deal with it in ways that we can. We do not choose or take problems as if they were lying around "out there" in the environment objective and independent of our actions: we bring them forth, we pose them.

> The most important ability of all living cognition is precisely, to a large extent, to pose the relevant questions that emerge at each moment of our life. They are not predefined but *enacted*, we *bring* them *forth* against a background, and the relevance criteria are oriented by our common sense, always in a contextualized fashion (Varela, 1996, 91, our translation).

The problems that we encounter and the questions that we ask are thus as much a part of us as they are a part of our environment since they emerge from our interaction with it. We interpret events as issues to address: we see them as problems to solve. We are not acting on pre-existing situations: our co-determination and continual interaction with the environment creates, enables and specifies the possible situations for us to act upon. The problems that we solve are then relevant for us, because we allow these to be problems for us while the environment "triggers" them in us. Some "issues" of the environment that would "trigger" elements in some persons do not "trigger" the same elements in others. The effects of the environment are not in the environment: they are made possible by the organism in constant/continual *interaction* with/in its environment.

Hence we claim that reactions to a prompt do not reside inside either the knower or the prompt: they emerge from the knower's *interaction* with the prompt, through posing what is relevant in the moment. If one adheres to this perspective for (mathematics) teaching and learning, one cannot assume, as René de Cotret (1999) notes, that instructional properties are present in the (mathematics) prompts offered and that these properties will determine learners' reactions. Strategies for solving problems emerge in the interaction of the knower and the problem, enacted in the interaction (Bautista, Roth, 2012; Maheux, Roth, 2011; Thom *et al.* 2009), influenced by the task but determined by the knower's experiences in solving similar and different problems: in his/her solving habits for similar or different problems, in his/her successes in with specific approaches, in his/her understanding of the problems, etc. Or, as Davis (1995) explains, (mathematical) strategies for solving are inseparable from the knower *and* from the problem itself, emerging from both, being "new" to some extent, dependent on/ influenced by the problem and its context, but determined by the learner with regard to his/her experiences: one's own complex histories and situations (Davis, Sumara, Kieren, 1996). Strategies are thus not predetermined, but continually generated for solving problems, emergent in the interaction with the problem when the solver engages and evolves with/in it. Threlfall (2002) explains that regarding solving mental mathematics problems:

> As a result of this interaction between noticing and knowledge each solution 'method' is in a sense unique to that case, and is invented in the context of the particular calculation – although clearly influenced by experience. It is not learned as a general approach and then applied

to particular cases. The solution path taken may be interpreted later as being the result of a decision or choice, and be called a 'strategy', but the labels are misleading. The 'strategy' (in the holistic sense of the entire solution path) is not decided, it emerges (*Ibidem*, 42).

In sum, strategies for solving emerge in the interaction of knower and prompts, where the knower plays an important role as he or she poses the problems, and where the nature of the problem plays a role as well, with a strategy tailored to it. In that sense, to build on our previous work (Simmt, 2000), problems given are not problems but prompts for solvers to create problems with: *prompts are offered, not problems*. Problems become problems when knowers engage with them, when they pose them as problems to solve. Thus knowers *transform* the prompt into mathematical problems for themselves, making the problem their own, which is often different from the designer's intentions (René de Cotret, 1999). By doing this, knowers enact their knowing by generating a strategy, one tailored to the problem (they) posed. In this sense, the posing is emergent as well as its solving. Acts of posing and solving are not predetermined but are generated in interaction with prompts, and influence one another as acts of posing influence the strategies enacted, which also modifies the problem being solved, and so forth. This has significant repercussions concerning education, which we address in the conclusion.

Returning to the case of the parents and children working mathematically together, we note that the facilitator provided a prompt and some materials with which to work. But it was the learners that defined their task; the problems emerged moment by moment. For one pair this was a geometric task, for the other a function task. Although the mathematics in which they engaged differed between the pairs, both pairs were observed to be engaged in what we view as mathematical activity: observing, recording, symbolizing and reflecting on patterns generated through functional relationships; observing, recording and working geometrically; conjecturing; and reasoning inductively and deductively.

This said, observing these knowers from an enactivist perspective, we find ourselves troubled by assumptions about these participants being said to acquire knowledge, where knowledge is seen as a thing to acquire, since all of our observations point to these learners who are not only doing mathematics but are being mathematical. We now turn to this issue in detail.

Theme 3: Knowledge as acquisition and knowledge as being

Several mathematics education scholars have drawn on enactivist ideas to rethink what it means to know mathematically and to reflect on mathematics knowledge. Focusing on emergence, adaptation and inseparability/co-specification of knowers and their environment, mathematical cognition has been defined as a dynamic process that emerges in action and interaction with the environment (Pirie, Kieren, 1994) rather than being seen in terms of mental representations that individuals construct in their minds of external phenomena found in the environment.

> [Radical constructivism] starts from the assumption that knowledge, no matter how it be defined, is in the heads of persons, and that the thinking subject has no alternative but to construct what he or she knows on the basis of his or her own experience (Glasersfeld 1995, 1).

This constrast with the enactivist view where:

> [Cognition] is something that is normally seen to be done with others. In [the] classroom, children's mathematics occurred along with that of their peers. [...] An enactivist view suggests that cognising, thinking, or doing mathematics with others in some way is the norm. Of course, human beings think mathematics for themselves but this thinking is done, at least in anticipation of, communicating with others and acting in a community of others interested in mathematics (Kieren, 1995, 7).

From an enactivist perspective, knowledge is not a thing, but is the interaction with the environment that can be observed by another (which could be oneself) as adequate action in a particular domain. As discussed in the previous section, such interaction brings forth the knower and the knower's world of significance (Kieren, Simmt, 2009). This view comes from Maturana's (1987) understanding of knowledge as *adequate action*, adequate conduct, in the context of its emergence. In other words, knowledge is an adequate response that *fits* in the context in which it emerges and fits with the "learner's history of experiences in the domain". Of particular significance to this view is that knowledge is enacted, with/in the meeting with the environment. Thus, for Maturana, knowledge is not something that one "possesses":

I am saying, "Knowledge is *never* about something. Knowledge is being in some manner". Yes, but I want you to listen to the conceptual shift involved in what I say. If I say I am a musician in the standard way, I mean that I know about music, that I can say things about such a thing that is *there*, independent of me, which is music, because I conceive knowledge as knowing *about* something. In our usual view of knowledge there must be a content to it, something that knowledge somehow embraces and reveals. What I am saying, however, is something completely different. I am saying that knowledge is never about something. I am saying that knowledge is adequate action in a domain of existence, that knowledge is a manner of being, that knowledge has no content because knowledge *is* being (Maturana, in Simon, 1985, 37).

(Mathematical) *knowing* in this perspective is inseparable from (mathematical) *doing* (Davis, 1996). "Knowledge is not in the book or in the library; Knowledge is not in our heads; knowledge is in the inter-action." (Kieren, Calvert, Reid, Simmt, 1995), "found in the actions by which the organism coordinates with its surrounding conditions" (Maheux, Roth, 2011, 36). All doing is knowing, all knowing is doing, as Maturana and Varela (1992) would say. This view of knowledge-as-action (Bateson, 1972), as an emergent adapted response, alters how (mathematical) knowing is perceived, shifting in turn the focus of attention when studying students' (mathematical) knowing.

Said more strongly, (mathematical) strategies put forth by knowers to solve problems, their adapted responses, are not to be seen as illustrations of their knowing, but *are* their knowing: the process of knowing and its product are one and the same thing (Pirie, Kieren, 1994). The adaptation process required to engage in a problem is not a representation of one's capacity for knowing, but *is* one's knowing: adaptation and action *are* knowledge. In that same sense, knowers' engagements in problems, as they pose them for solving them, is not an illustration of how they know, *but is* what they know, how they are, and who they are.

So, what do people point to when they claim knowledge? Where is it? First, let us be reminded of Maturana's maxim, that "Everything is said by an observer" (Maturana, 1987). It is the observer who recognizes, who reifies, what he or she sees as being knowledge. Thus, if I want to know if someone knows to play piano, I give him/her a piano and I observe. The same, as Maturana explains, for algebra:

Thus, if someone claims to know algebra – that is, to be an algebraist – we demand of him or her to perform in the domain of what we consider algebra to be, and if according to us she or he performs adequately in that domain, we accept the claim (Maturana, 1988b, 4-5).

This quote raises for us another important point, about the observer: "Knowledge is adequate action in a domain specified by a *questioner*" (Maturana, in Simon, 1985, 37, our emphasis). So, when working from an enactivist perspective, pointing to something as adequate is pointing to the action and interaction, and not to an objectively fixed referent. It is an act of observation that the observer, who judges, has made based on his or her own set of criteria. The action and interaction is not taken "as-is", but as an act of distinction that determines that object of observation. This access to knowledge that enactivism suggests is particularly significant for us, researchers in mathematics education, as we attempt to appreciate the mathematical activity of students: activity which turns out to be observations, conceptualizations made by us, observers (Maheux, 2010) and that we assign to/impose on the observed.

But, this change from knowledge as object, as possession is an important one for mathematics educators and educational researchers, and it poses significant challenges. This view is not new, however, as many theorists have attempted to think in different ways about knowledge for number of years. But this view needs to be explored further, especially under an enactivist lens. Under different theorizations, we have been and still are talking in terms of knowledge. One question we might ask, as we did in Maheux, Proulx (2012) is: "Could it be that we need to drop our usage of the word knowledge? And, if so, replace it by what?". Exploring this avenue imposes new sets of ideas, new views, etc., which are still to be developed.

In mathematics education, one possible way of exploring this conceptualization is through thinking in depth about the differences that Davis, Sumara and Luce-Kapler (2008, 23) establish between *mathematics* and *mathematical*.

"Mathematics" is generally used to refer to a body of knowledge – that is, a widely accepted collection of concepts and procedures that have emerged through centuries of inquiry. […] it is a domain that continues to grow and evolve. "Mathematical", in contrast, is more a reference to

a mode of thinking. It involves a noticing of sameness and difference, of pattern and irregularity, of specifics and generalizations, of abstract principles and concrete objects. "Mathematical", true to the meaning of its ancient Greek root *manthanein*, is about learning.

When we refer to a person's mathematics we are making a claim about something that this person "has" whereas when we refer to a person as mathematical our claim is about their "being". When observing learners engaged in mathematics and saying "the learner knows mathematics", what the observer sees is the "mathematical". In so far as mathematics knowledge has been conceptualized as a possession, a thing that one can have, then mathematics can be acquired, stored, accessed, and used. However, if one sees knowledge as something that ones does, that one enacts, then the learner is seen as one who mathematizes. However, if knowing mathematics is to *be* mathematical, then we as observers distinguish the knower as bringing forth a world of mathematical significance and the knower as a mathematical being. The following Table 1 offers a view of the distinctions that can be traced from mathematics as something to acquire, as something one does, to something one enacts.

Acquire acquisition metaphors	HAVE	Mathematics	Known
Act action metaphors	DO	Mathematize	Know
Identify identity metaphors	BE	Mathematical	Knower

Table 1. Distinctions among knowledge, knowing and knower

In our example with the mother and daughter we see them engage in actions we as observers distinguish as geometrical. Manipulating shapes through space, reflecting and transforming shapes, and making conjectures about placement and order all speak to this pair's affinity for that particular kind of activity. It points to their being with shapes and numbers, arrangements and counts and it points to who they are. As observers of these two parent-child pairs we make distinctions between their actions, their ways of being with the prompt and materials. In so far as the mother and daugh-

ter were geometrical in their actions, the father and daughter were much more functional (in the sense of a relationship between two variables). They "knew" through number patterns and relationships in the growth of how many arrangements. For them being mathematical was being relational. It makes little sense to speak of the geometry or the function that these parents and children acquired or used while doing the task without speaking about them. All of their actions suggest that the materials and the prompt triggered their identities as mathematics knowers leading them to mathematize the problems that emerged for them and leaving them not with some mathematics but with having experienced themselves as mathematical.

Concluding remarks

> In short: the world is not something that is given to us but something we engage in by moving, touching, breathing, and eating. This is what I call *cognition as enaction* since enaction connotes this bringing forth by concrete handling (Varela, 1999, 8).

So what, then? We have offered, in this article, three themes from enactivism (there are others yet to be probed into) that we as mathematics educators and researchers see as significant to begin conceptualizing learning and knowledge differently. It is our contention that enactivism has enabled us as observers to think differently about mathematical understanding. By paying attention and recognizing the distinction between constructivism and enactivism, we have shifted from thinking about: a person's interpretation of the world to their bringing forth a world of significance; to cognition as problem-solving to cognition as problem-posing; to knowledge as an acquisition, to knowing as action, to knowing as a way of being. These enactivist themes shape the theory from which we work. They frame our view of knowledge, learning and meaning making. Therefore, they have repercussions on and for our understanding of curriculum and pedagogy. In particular, we believe that a significant implication of this enactivist frame is that it compels us to educate differently and towards different ends. Like Davis *et al.* (2008), we "do not see education in linear-causal terms of achieving preset objectives or re-presentation of established truths, but as a participation in the ever-unfolding project of becoming capable of new,

perhaps as-yet unimaginable possibilities (*Ibidem*, 20-21). To educate students is not anymore to ensure they "have" mathematics, but it is that they "become mathematical. In that sense, the goal of education is less about acquiring things, and more about becoming some-thing. Whereas

> in a typical mathematics class, the temptation […] have been to focus on the final products of the students' efforts – i.e. their symbolic representations and their logical arguments – in order to assess the appropriateness of their actions and to judge the worth of the activity Enactivism prompts us to attend as closely to the preceding actions – the unformulated exploration, the undirected movement, the unstructured interaction, wherein the body is wholly engaged in mathematical play – as to the formal mathematical ideas that might emerge from those actions (Davis, Sumara, Kieren, 1996, 156).

Towers, Martin and Heater (2012) infer, based on Maturana (Maturana, Poerksen, 2004), that what students learn when they interact with each other and a teacher in a mathematics classroom is primarily a way of living (with mathematics) and that it is only through such a process that specific knowledge is developed. In other words, specific mathematics knowledge (the very thing that most teachers are trying to teach) is a kind of by-product of this other, more significant, process. This prompts us to aim at directing attention to the ways of being that are being fostered in classrooms (and not to monitoring the knowledge generated).

The views we offer here, grounded in what we label as enactivist, are not to be seen as the thing-in-themselves, but mainly as matters of reflections, for thinking differently, for pushing the boundaries about what we understand about knowing. It is an intention to take the work that the constructivist discourse has established, and continue to push it further. Enactivism for us plays this role, offering possibilities for thinking differently, for apprehending known concepts and ideas under a new and revitalizing lens, for knowing more about knowing or, as von Foerster (2003) would have it, for understanding understanding.

Author's Presentation: Jérôme Proulx is Professor of "Mathematics Education" at University of Québec - Montréal (Canada).
Elaine Simmt is Professor of "Mathematics Education & Director", Master of Education in Educational Studies University of Alberta (Canada).

Notes

[1] Our entry into enactivism has been done as mathematics education researchers and thus our examples come from it. We see them, however, as pertinent for other areas of education.

References

Bateson, G. (1972), *Steps to an ecology of mind,* New York, Ballantine Books.

Bautista, A., Roth, W.M. (2012), "Conceptualizing sound as a form of incarnate mathematical consciousness", *Educational Studies in Mathematics*, *79*, 1, 41-59.

Bednarz, N., Proulx, J. (2011), "Ernst von Glasersfeld's contribution and legacy to a didactique des mathématiques research community", *Constructivist Foundations*, *6*, 2, 239-247.

Capra, F. (1996), *The web of life: A new understanding of living systems*, New York, Anchor Books.

Davis, B. (1995), "Why teach mathematics? Mathematics education and enactivist theory", *For the Learning of Mathematics,* 15, 2, 2-9.

— (1996), *Teaching mathematics: Toward a sound alternative*, New York: Garland.

— Sumara, D., Kieren, T. (1996), "Cognition, co-emergence, curriculum", *Journal of Curriculum Studies*, *28* (2), 151-169.

— Sumara, D., Luce-Kapler, R. (2000), *Engaging minds: learning and teaching in a complex world*, Mahwah, NJ, Lawrence Erlbaum.

Foerster, H. von (2003), *Understanding understanding*, New York, Springer-Verlag.

Glasersfeld, E. von. (Ed.) (1991), *Radical Constructivism in Mathematics Education*, Dordrecht, Kluwer Academic Publishers.

— (1992), Aspects of Radical Constructivism and its Educational Recommandations. Paper presented at ICME-7, in the *Working Group #4*, Quebec, Canada.

— (1995), *Radical Constructivism: A Way of Knowing and Learning*, London and Washington, Falmer Press.

Kieren, T.E. (1995, June), *Teaching mathematics (in-the-middle): Enactivist view on learning and teaching mathematics*, Paper presented at the Queens/Gage Canadian National Mathematics Leadership Conference, Queens University, Kingston, Canada.

Kieren, T., Simmt, E. (2009), "Brought forth in bringing forth: the inter-actions and products of a collective learning system", *Complicity: An International Journal of Complexity and Education 6*(2), 20-27.

Kieren, T., Calvert, L.G., Reid, D., Simmt, E. (1995), "Coemergence: Four enactive portraits of mathematics activity", Presentation given at the 1995 AERA conference, San Francisco, CA, http://www.eric.ed.gov/ERICWebPortal/contentdelivery/servlet/ERICServlet?accno=ED390706.

LAROCHELLE, M., BEDNARZ, N., GARRISON, J. (Eds.) (1998), *Constructivism and education*, Cambridge, Cambridge University Press.

MAHEUX, J.F. (2010), *How do we know? An epistemological Journey in the Day-to-day, Moment to-moment, of Researching, Teaching and Learning in Mathematics Education*, Ph.D thesis, University of Victoria.

MAHEUX, J-F., ROTH, W-M. (2011), "Relationality and mathematical knowing", *For the Learning of Mathematics*, *31*(3), 36-41.

MATURANA, H. (1987), "Everything is said by an observer", in W.I. Thompson (Ed.), *Gaia A Way of Knowing: Political Implications of the New Biology*, Hudson, NY, Lindisfarne Press.

— (1988a), "Reality: The search for objectivity of the quest for a compelling argument", *Irish Journal of Psychology*, *9* (1), 25-82.

— (1988b), "Ontology of observing: The biological foundations of self consciousness and the physical domain of existence", *Texts in cybernetic theory, conférence workbook*, American Society for Cybernetics, Felton, California.

MATURANA, H.R., POERKSEN, B. (2004), *From being to doing: the origins of the biology of cognition*, Allemagne, Carl-Auer Verlag.

MATURANA, H.R., VARELA, F.J. (1992), *The tree of knowledge: The biological roots of human understanding* (Rev. Ed.), Boston, MA, Shambhala.

NODDINGS, N. (1990), "Constructivism in mathematics education", *Journal for Research in Mathematics Education*, Monograph # 4, 7-21.

PHILIPPS, D.C. (Ed.). (2000), *Constructivism in education: Opinions and second opinions on controversial issues*, Chicago, National Society for the Study of Education.

PIRIE, S., KIEREN, T. (1994), "Growth in mathematical understanding: How can we characterize it and how can we represent it? ", *Educational Studies in Mathematics, 26*(2-3)*, 165-190.

PROULX, J. (2006), "Constructivism: A re-equilibration and clarification of the concepts, and some potential implications for teaching and pedagogy", *Radical Pedagogy*, *8*(1). http://radicalpedagogy.icaap.org

— (2008a), "Elaborations on Maturana and Varela's theory of cognition: Some introductory notes on how this perspective differentiates itself from constructivist theories", *Complicity: An international journal of complexity and education*, *5*(1), 11-26.

— (2008b), "Structural determinism as hindrance to teachers' learning: Implications for teacher education", *Proceedings of PME-32, 4*, 145-152, México, PME.

— (2010), "Is "facilitator" the right word? And on what grounds? Some reflections and theorizations", *Complicity: An International Journal of Complexity and Education*, *7*(2), 52-65.

PROULX, J., MAHEUX, J-F. (2012), "Épistémologie et didactique des mathématiques: questions anciennes, nouvelles questions", *For the Learning of Mathematics*, *32* (2), 41-46.

PROULX, J., SIMMT, E., TOWERS, J. (2009), "Enactivism in mathematics education", In TZEKAKI, M., KALDRIMIDOU, M. SAKONIDIS, C. (Eds.). *Proceedings of*

the 33rd Conference of the International Group for the Psychology of Mathematics Education, Thessaloniki, Greece, PME, Vol. 1, 1-249 – 1-252.

REID, D. (1996), "Enactivism as a methodology", *Proceedings of PME-20*, 4, 203-210, Valencia, Spain, PME.

RENÉ DE COTRET, S. (1999), "Perspective bio-cognitive pour l'étude des relations didactiques", in G. LEMOYNE, F. CONNE (Eds.), *Le cognitive en didactique des mathématiques*, Montréal, Qc, Presses de l'Université de Montréal, 103-120.

SIMMT, E. (2000), *Mathematics knowing in action: a fully embodied phenomenon*, Unpublished doctoral dissertation, University of Alberta, Canada.

— (2006), *Complexity and mathematics education: discussion paper*, Paper presented at the 30th conference of the International Group for the Psychology of Mathematics Education, Prague, Czech Republic.

SIMON, R. (1985), "A frog's eye view of the world. Structure is destiny: An interview with Humberto Maturana", *The Family Therapy Networker, 9* (3), 32-37, 41-43.

STEFFE, L., GALE, J. (Eds.), (1995), *Constructivism in education*, Hillsdale, NJ, Lawrence Erlbaum Associates.

THOM, J.S., NAMUKASA, I.K., IBRAHIM-DIDI, K., McGARVEY, L.M. (2009), "Perceptually guided action: Invoking knowing as enaction", in J. PROULX, E. SIMMT, J. TOWERS, RF05: The enactivist theory of cognition and mathematics education research: Issues of the past, current questions and future directions, *Proceedings of PME-33*, Thessaloniki, Greece, vol. 1, 249-278.

THRELFALL, J. (2002), "Flexible mental calculations ", *Educ. Studies in Mathematics, 50*, 29-47.

TOWERS, J., MARTIN, L., HEATER, B. (submitted), *Teaching and Learning Mathematics in the Collective*, Manuscript submitted for publication.

VARELA, F. J. (1996), *Invitation aux sciences cognitive*, Paris, Éditions du Seuil.

— (1999). *Ethical know-how: action, wisdom, and cognition,* Stanford, Stanford University Press.

— THOMPSON, E., ROSCH, E. (1991), *The embodied mind: cognitive science and human experience*, Cambridge, The MIT Press.

— POERKSEN, B. (2004), "Truth is what works", in B. POERKSEN (Ed.), *The certainty of uncertainty: Dialogues introducing constructivism*, Imprint Academic, UK, 85-107.

WATZLAWICK, P. (Ed.) (1984), *The invented reality: How do we know what we believe we know? Contributions to constructivism*, New York, Norton.